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Extension of the Quantum Mechanical Principle of Superposition to Non-Identical States with Short Life Times[1]

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We apply the principle of superposition to two quantum mechanical states of which one has a short life time $1/\Gamma$. They must be identical, but may have a small difference in energy-momentum comparable to Γ. This produces new effects resulting from the linear relationship that appears between the S-matrix element and the production cross section of the decay products of the short lived state. An optical theorem for diffraction dissociation processes is proposed.

This mechanism also provides a measure of the non-orthogonality between unstable particles that are eigenstates of a non-Hermitian Hamiltonian.

KEY WORDS: quantum mechanics; superposition principle; optical theorem; unstable states; diffractive scattering.

1. INTRODUCTION

In the field of hadron-hadron interactions, there is a type of process known as 'quasi-diffractive' (See, e.g., Amaldi *et al.*, 1976; Leith, 1975) where an incoming particle is excited to a short-lived state (which subsequently decays). These processes have a surprisingly high cross-section not unsimilar to diffractive scattering proper where the incoming particles remain the same and where the incoming and the outgoing states are in quantum mechanical interference, causing the cross-section to be very large.

Should it not be possible to make the same principle also work for the quasi diffractive processes? Then we would have to have quantum mechanical interference not only between 'identical states' but also between 'similar states.' The following contribution is in the line of this idea.

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2. EXTENSION OF THE QUANTUM MECHANICAL PRINCIPLE OF SUPERPOSITION TO NON-IDENTICAL STATES WITH SHORT LIFE TIMES

2.1. Present Situation—Interference for Identical States

In a two-body reaction between stable states

$$a + b \longrightarrow c + d \tag{1}$$

one distinguishes the general inelastic channel from the special elastic channel in which $c = a$ and $d = b$. In terms of a scattering amplitude $f(\theta)$ in the center of mass system, the differential cross section is related to the amplitude in both cases by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{2}$$

Despite the apparent similarity in the form of (2), the elastic case is quite special because its amplitude contains the incoming beam as an additive term, on account of the identity of the incoming and the outgoing states. This can best be seen in a partial wave expansion of $f(\theta)$ (See, e.g., Blatt and Weisskopf, 1965; Roman, 1965).

$$f^{el}(\theta) = \frac{1}{2ik}\sum_l (2l+1)\big(S_l^{el} - 1\big)P_l(\cos\theta) \tag{3}$$

$$f^{\text{inel}}(\theta) = \frac{1}{2ik}\sum_l (2l+1)S_l^{\text{inel}}P_l(\cos\theta) \tag{4}$$

Here the S_l are the matrix elements of the lth partial wave unitary S-matrix. Let us consider a number of channels of reaction 1, let the elastic channel be denoted by the ket $|1\rangle$, the first inelastic channel by $|2\rangle$, another one by $|3\rangle$, etc. Then Eqs. 3 and 4 may be written as a transition (index l suppressed):

$$|1\rangle \to |f\rangle = S_{11}|1\rangle - |1\rangle + S_{12}|2\rangle + S_{13}|3\rangle + \cdots \tag{5}$$

The cross section is proportional to

$$\begin{aligned}\langle f|f\rangle &= |S_{11}|1\rangle - |1\rangle + S_{12}|2\rangle + S_{13}|3\rangle + \cdots|^2\\ &= |S_{11} - 1|^2\langle 1|1\rangle + |S_{12}|^2\langle 2|2\rangle + |S_{13}|^2\langle 3|3\rangle\cdots\end{aligned} \tag{6}$$

and terms proportional to $\langle 1|2\rangle$ etc., referring to different final states, are zero due to their orthogonality, because they show no interference, they are incoherent. In order to interfere they would have to be identical in all respects. Only the incoming wave $|1\rangle$ and the scattered wave $S_{11}|1\rangle$ are identical in all respects and interfere with each other; this is expressed in the notation where the same ket is attached to both waves.

In particular the momenta of two states have to be equal if they are to interfere. This is intuitively evident—consider two plane waves ψ_1 and ψ_2 propagating along z. Their superposition would give a probability P along z

$$P(z) = |\psi_1 + \psi_2|^2 = |e^{ik_1 z} + e^{ik_2 z}|^2 \tag{7}$$

The probability oscillates and the interference term averages out to zero if $k_1 - k_2 \neq 0$: for the actual detection in a macroscopic detector extending from $z = -a$ to $z = a$ the measured rate can be written as

$$\frac{1}{2a}\int_{-a}^{a} P(z)\,dz = 2 + \frac{1}{2a}\int_{-a}^{a} 2\mathrm{Re}\, e^{i(k_1-k_2)z} dz$$
$$= \begin{cases} 2 & \text{if } k_1 \neq k_2 \text{ (i.e. } (k_1 - k_2)a \gg 1) \\ 4 & \text{if } k_1 = k_2 \text{ (i.e. } (k_1 - k_2)a \ll 1) \end{cases} \tag{8}$$

The two momenta have to be equal for interference to take place.

2.2. Proposal—Interference for Similar States if One has Short Life Time

Now we introduce short lived states. Let particle c in reaction 1 be unstable with mean life $1/\Gamma$ and some decay products $x, y \ldots$ Furthermore, let b and d be identical. As an example we consider the amplitudes for the two processes

$$p + p \longrightarrow p + p \tag{9}$$

$$p + p \longrightarrow N^* + p \tag{10}$$

(subsequently $N^* \longrightarrow p + \pi \ldots$). The N^* is to have the same quantum numbers as the proton, except for a different mass. The difference in mass results in a difference in momenta of p and N^*. We propose that interference takes place between the two processes (9) and (10) and between the incoming proton beam and the outgoing N^* in (10). In this way we extend the principle of superposition from cases of 'complete identity' to cases of 'approximate identity' which we define in a first step as 'identical in all respects except for a small difference in energy-momentum comparable to Γ.'

The reason presented above for the absence of interference does not apply because the averaging out is no longer complete if one of the two amplitudes is quickly decaying: as the phase difference is increasing towards 180 degrees and the interference term changes sign the amplitude has already decreased, and the compensation will be incomplete, c.f. Fig. 1 for an illustration. With the appearance of unstable states, the macroscopic scale a as a measure of coherence length required for interference is replaced by a microscopic measure at the scale of Γ.

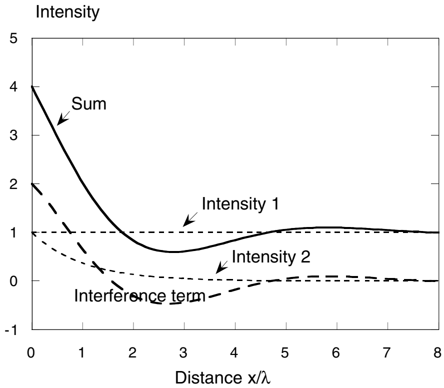

Fig. 1. Interference of one stable and one decaying amplitude that start out at $x = 0$ with equal size and phase. With the superposition principle applied, the intensity is $|e^{ik_1x} + e^{i(k_2+i\kappa)x}|^2 = 1 + e^{-2\kappa x} + 2\mathrm{Re}\, e^{i(k_1-k_2)x-\kappa x}$. The values chosen were $k_1 - k_2 = 1/\lambda$ and $\kappa = 0.5/\lambda$ leading to a yield ratio [c.f. Eq. 21] of $Y_{l/q} = 0.4$.

2.2.1. Short Review of the Space-Time Description of Unstable States

As explained in another contribution to this volume (Blum and Saller, 2004) an unstable particle with given energy-momentum $p := (E, \vec{k})$ and mean life time $1/\Gamma$ is characterized by a small four vector (the 'spread vector') $q := (\Delta E, \overrightarrow{\Delta k})$ which is spacelike, in contrast to the timelike p.

There are three Lorentz invariants:

$$\begin{aligned} p^2 &= E^2 - (\vec{k})^2 = m^2 \\ pq &= E\Delta E - \vec{k}\overrightarrow{\Delta k} = m\Gamma/2 \\ q^2 &= (\Delta E)^2 - (\overrightarrow{\Delta k})^2 = -B^2 \leq 0 \end{aligned} \tag{11}$$

where m is the rest mass, and B^2 (capital beta) is an invariant which contains information from the kinematic conditions of production, like masses and momenta of the particles in the production process. $|B|^2$ is typically of the order of $\Gamma^2/4$ but may also be very small or zero when the particles co-produced with the unstable particle in question have a total invariant mass that is small compared to the total energy of the reaction or that is zero.

There is a special Lorentz frame (the 'sharp energy system') for which $\Delta E = 0$, and another one (the 'central rest system') for which $\vec{k} = 0$. But there is no Lorentz frame for which $\overrightarrow{\Delta k} = 0$, in other words: there is no Lorentz frame in which the unstable particle comes entirely to rest ('entirely' referring to the statistical distribution of invariant mass—if the peak of the Breit-Wigner distribution is at rest, the parts in the wings run away in opposite directions).

In the most general case there are 8 real numbers to define the energy-momentum properties of an unstable state, and the vectors $\vec{k}$ and $\overrightarrow{\Delta k}$ do not have to be parallel. But many situations are simpler. For example, an unstable particle produced in a two-body reaction is characterized by two real numbers in addition to its energy-momentum, if one is in the c.m. system of the reaction. And there is a sharp energy system where only one real number characterizes the decaying particle in addition to the 4 numbers of energy-momentum.

2.2.2. Short Review of the Quantum Mechanical Description of Unstable States

It can be shown (Blum and Saller, 2003) that in a relativistically invariant wave function for a stable particle such as

$$\psi(t, \vec{x}) = e^{i(Et - \vec{k}\vec{x})} \tag{12}$$

or

$$\psi(t, r) = \frac{1}{r} e^{i(Et - kr)} \tag{13}$$

the effect of instability may be introduced in a relativistically compatible form by using the spread vector $(\Delta E, \overrightarrow{\Delta k})$ described in subsection 2.2.1. One replaces in Eqs. (12) or (13) the real by the complex four vector

$$E \longrightarrow E + i\Delta E \quad \text{and} \quad \vec{k} \longrightarrow \vec{k} + i\overrightarrow{\Delta k} \tag{14}$$

and obtains for $t > 0$ in first order of Γ/m the wave functions

$$\psi(t, \vec{x}) = e^{i(E + i\Delta E)t - i(\vec{k} + i\overrightarrow{\Delta k})\vec{x}} \tag{15}$$

or

$$\psi(t, r) = \frac{1}{r} e^{i(E + i\Delta E)t - i(k + i\Delta k)r} \tag{16}$$

Equation (16) describes a spherical wave where the momentum spread is parallel to the momentum. In the 'sharp energy' Lorentz frame, $\Delta E = 0$, and Eq. (16) takes the form

$$\psi(t, r) = \frac{1}{r} e^{iEt - i(k - i\kappa)r} \tag{17}$$

where the spatial damping Δk assumes the value

$$\kappa = \frac{m\Gamma}{2k} \tag{18}$$

The appearance of complex momenta together with complex energies is a consequence of the fact that they are subject to Lorentz transformations. The imaginary parts, being differences of energy-momentum, transform as 4-vectors (Blum, 2000; Blum and Saller, 2004). In a Lorentz system in which the imaginary part $i\Delta E$ of the spacelike $(i\Delta E, i\Delta k)$ vanishes, it is the complex momentum that describes the decay.

2.2.3. Superposition Principle Applied

For the treatment of short lived particles it is appropriate to consider spherical waves with their well defined production point. Let ψ_1 be a stable spherical wave with momentum k_1 and ψ_2 an unstable spherical wave with complex momentum $k_2 + i\kappa$, and let them interfere. We have a situation of continuous flow, and let the energies of ψ_1 and ψ_2 be equal and real in the sharp energy system; then $k_1^2 - k_2^2 = m_2^2 - m_1^2$. We should later describe more general situations. For the probability density one obtains

$$P(r) = |\psi_1 + \psi_2|^2 = \left| a_1 \frac{e^{ik_1 r}}{r} + a_2 \frac{e^{i[k_2 + i\kappa]r}}{r} \right|^2$$
$$= \frac{|a_1|^2}{r^2} + \frac{|a_2|^2}{r^2} e^{-2\kappa r} + 2\mathrm{Re} a_1 a_2^* \frac{e^{i(k_1 - k_2)r - \kappa r}}{r^2} \tag{19}$$

Integration over the volume of a sphere with radius R $\gg 1/\kappa$, $1/(k_1 - k_2)$ produces an expression for the number of particles present in this sphere around the target. (In the second and third terms the integration may be extended to ∞.)

$$\int_0^R P(r) 4\pi r^2 dr = 4\pi |a_1|^2 R + 4\pi \frac{|a_2|^2}{2\kappa} + 2\mathrm{Re} \frac{4\pi a_1 a_2^*}{-i(k_1 - k_2) + \kappa}$$
$$= 4\pi |a_1|^2 R + 4\pi \frac{|a_2|^2}{2\kappa} + 8\pi |a_1||a_2| \mathrm{Re} \left(e^{i\phi} \frac{\kappa + i(k_1 - k_2)}{(k_1 - k_2)^2 + \kappa^2} \right) \tag{20}$$

where ϕ is the phase difference between a_1 and a_2. The first term is the pure component of the stable state, term 2 is the pure component of the unstable state, and term 3 the component consisting of both states. The decay products are created by the decay of terms 2 and 3, but the rate of decay of the interference term is only half the value from term 2. We may form the ratio of the decay rates of both components and call it the yield ratio $Y_{l/q}$ of the linear term 3 over the quadratic term 2.

$$Y_{l/q} = \frac{|a_1|}{|a_2|} 2\mathrm{Re} \left(e^{i\phi} \frac{\kappa^2 + i(k_1 - k_2)\kappa}{(k_1 - k_2)^2 + \kappa^2} \right) \tag{21}$$

It vanishes as it should when $\kappa = 0$, and it becomes large when the amplitude ratio $|a_2|/|a_1|$ becomes small.

In the notation of Eq. (5) we identify the proton with $|1\rangle$, the N^* with $|2\rangle$, whereas $|3\rangle$ is any other channel that is not interfering. Instead of (6) we now have

$$\langle f|f\rangle = |S_{11}-1|^2\langle 1|1\rangle + 2\mathrm{Re}[(S_{11}-1)S_{12}^*\langle 2|1\rangle] + |S_{12}|^2\langle 2|2\rangle + |S_{13}|^2\langle 3|3\rangle \ldots \tag{22}$$

The presence of the term $\langle 2|1\rangle$ means that a state is created that is undecided between $|1\rangle$ and $|2\rangle$; it decays and liberates decay products. As $\langle f|f\rangle$ is proportional to the cross section of the partial wave, Eq. (22) establishes a new relation between the production cross section of the decay products of $|2\rangle$ and the S-matrix element S_{12}. This relation contains a quadratic as well as a linear term of S_{12}.

In a comparison between Eqs. (22) and (19), the notation of (22) is more general: The term $\langle 2|1\rangle$ is the degree of non-orthogonality between $|1\rangle$ and $|2\rangle$ whereas the interference term in Eqs. (19) to (21) is the degree of non-orthogonality due to the decay and the difference of momenta alone. There could be other factors such as internal quantum numbers to reduce the total non-orthogonality. We compare Eqs. (21) and (22), equating $(S_{11}-1) = a_1$ and $S_{12} = a_2$ and obtain

$$2\mathrm{Re}\left(\frac{a_1 a_2^*}{|a_2|^2}\frac{\langle 2|1\rangle}{\langle 2|2\rangle}\right) = Y_{l/q} = 2\mathrm{Re}\left(\frac{a_1 a_2^*}{|a_2|^2}\frac{\kappa^2 + i(k_1-k_2)\kappa}{\kappa^2+(k_1-k_2)^2}K\right) \tag{23}$$

This relation holds for any a_1, a_2, therefore

$$\frac{\langle 2|1\rangle}{\langle 2|2\rangle} = \frac{\kappa^2 + i(k_1-k_2)\kappa}{\kappa^2+(k_1-k_2)^2}K \tag{24}$$

where the complex number K $(0 \leq |K| \leq 1)$ describes these other factors. It could be called 'the degree of internal similarity' whereas the kinematic factor is the 'degree of external similarity.' The numerical value of (24), in first order of $m_2\Gamma/(m_2^2 - m_1^2)$, is

$$\frac{\langle 2|1\rangle}{\langle 2|2\rangle} = i\frac{m_2\Gamma}{m_2^2 - m_1^2}K$$

As a general orientation, we may imagine that for some quasi diffractive scattering process (with typically $m_2\Gamma/(m_2^2 - m_1^2) \approx 0.2$) at high energies, $|a_2|$ could be two orders of magnitude smaller than $|a_1|$, resulting in an amplification factor (23) of the cross section that would be two orders of magnitude, multiplied by the value of Eq. (24). Therefore a full order of magnitude for the amplification factor can reasonably be expected—albeit still depending on K.

2.3. Representation of Short Lived States in Hilbert Space

We have used the superposition principle on 'similar' or 'approximately identical' states, defined (in a first step) as 'identical in all respects except for a small difference in energy-momentum, comparable to Γ, the inverse life time of one of them.' This gave rise to a new relation between the S-matrix element for the production of the unstable state and the cross section for its decay products.

In Hilbert space, this implied the existence of non-zero transition elements between similar states, which are therefore no longer orthogonal to each other.

There is a theorem that Hermitian or self-adjoint Hilbert space operators have real eigenvalues and that the eigenstates are orthogonal to each other. If short lived states with complex energy-momentum are eigenstates of a Hamiltonian, it has to be a non-Hermitian Hamiltonian, and the non-orthogonality of similar states appears to be a natural consequence—which in turn justifies the extension of the superposition principle to 'similar' states.

The next step is to work out the norms of these states in Hilbert space. It turns out that a *collective norm* is appropriate *for the states that interfere with each other*. This is shown in the contribution of Heinrich Saller to this workshop.

3. OPTICAL THEOREM FOR DIFFRACTION DISSOCIATION

3.1. Recapitulation of the Optical Theorem we Know (Blatt and Weisskopf, 1965; Roman, 1965)

The amplitude $f(\theta)$ for the scattering angle θ is connected with the differential cross section by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{25}$$

In the partial wave expansion in Legendre functions $P(\cos\theta)$ we have for the elastic channel

$$f^{el}(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\big(C_l^{(1)}e^{2i\delta_l} - 1\big)P_l(\cos\theta) \tag{26}$$

and for every inelastic channel n ($n = 2, \ldots, N$)

$$f^{\mathrm{inel}}(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)C_l^{(n)}e^{2i\phi_l^{(n)}}P_l(\cos\theta) \tag{27}$$

Here the δ_l are the elastic phase shifts, the $\phi_l^{(n)}$ the production phases of the inelastic channels. The real numbers $C_l^{(n)}$ are between 0 and 1 and satisfy, for every l, the relation

$$\sum_{n=1}^{N}|C_l^{(n)}|^2 = 1, \tag{28}$$

which represents probability conservation in every partial wave.

For a calculation of the total cross section σ_{tot} it is easiest to distinguish the various channels by kets $|n\rangle$ that are orthogonal to each other. We write Eqs. (26) and (27) as

$$f(\theta)|f\rangle = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left[\left(C_l^{(1)}e^{2i\delta_l}-1\right)|1\rangle + \sum_{n=2}^{N} C_l^{(n)}e^{i\phi_l^{(n)}}|n\rangle\right] \tag{29}$$

For the complete σ_{tot} we have to form $|f(\theta)|^2\langle f|f\rangle$, integrate over all angles and sum over n. On account of the orthonormality conditions

$$\int d\Omega P_l(\cos\theta)P_l'(\cos\theta) = \frac{4\pi}{2l+1}\delta_{ll'} \tag{30}$$

$$\langle n|n'\rangle = \delta_{nn'} \tag{31}$$

one obtains

$$\begin{aligned}\sigma_{tot} &= \int d\Omega|f(\theta)|^2\langle f|f\rangle \\ &= \frac{\pi}{k^2}\sum_l(2l+1)\left(\left|C_l^{(1)}e^{2i\delta_l}-1\right|^2 + \sum_{n=2}^{N}\left(C_l^{(n)}\right)^2\right) \\ &= \frac{\pi}{k^2}\sum_l(2l+1)\left(2-2\mathrm{Re}\left(C_l^{(1)}e^{2i\delta_l}\right)\right)\end{aligned} \tag{32}$$

On the other hand, the elastic scattering amplitude for $\theta = 0$ is seen from (29) to be

$$f(0)|1\rangle = \frac{1}{2ik}\sum_l(2l+1)\left(C_l^{(1)}e^{2i\delta_l}-1\right)\left|1\right\rangle \tag{33}$$

using $P_l(1) = 1$ for every l. The imaginary part of $f(0)$ is

$$\mathrm{Im} f(0) = \frac{-1}{2k}\sum_l(2l+1)\mathrm{Re}\left(C_l^{(1)}e^{2i\delta_l}-1\right) \tag{34}$$

Comparing (34) and (32) we find the familiar optical theorem

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\mathrm{Im} f(0) \tag{35}$$

3.2. An Optical Theorem for Diffraction Dissociation

Now we introduce one diffraction dissociation channel capable of interference with the initial state; let $\langle 1|2\rangle \neq 0$. We compare the forward amplitude for the production of $|2\rangle$ with the cross section integrated over the angles:

$$f^{(2)}(0)|2\rangle = \frac{1}{2ik}\sum_l (2l+1)C_l^{(2)}e^{i\phi_l^{(2)}}|2\rangle \tag{36}$$

The indistinguishable states $|1\rangle$ and $|2\rangle$ have a production rate proportional to

$$|f^{(1)}(\theta)|^2\langle 1|1\rangle + |f^{(2)}(\theta)|^2\langle 2|2\rangle + f^{(1)}(\theta)f^{(2)*}(\theta)\langle 2|1\rangle \tag{37}$$

of which the second and third terms produce the decay products. Their cross section is

$$\begin{aligned}\sigma^{\text{decay products}} &= \int d\Omega(|f^{(2)}(\theta)|^2 + 2\mathrm{Re}[f^{(1)}(\theta)f^{(2)*}(\theta)\langle 2|1\rangle]) \\ &= \frac{\pi}{k^2}\sum_l (2l+1)\big(\big(C_l^{(2)}\big)^2\langle 2|2\rangle \\ &\quad + 2\mathrm{Re}\big[\big(C_l^{(1)}e^{2i\delta_l} - 1\big)C_l^{(2)}e^{-i\phi_l^{(2)}}\langle 2|1\rangle\big]\big)\end{aligned} \tag{38}$$

At high energies where the $C_l^{(1)}$ and $C_l^{(2)}$ are both $\ll 1$, we have in first order of the C_l

$$\sigma^{\text{decay products}} = \frac{\pi}{k^2}\sum_l (2l+1)2\mathrm{Re}\big(-C_l^{(2)}e^{-i\phi_l^{(2)}}\langle 2|1\rangle\big) \tag{39}$$

From (36) we see that in the forward direction

$$\frac{4\pi}{k}\mathrm{Re}f^{(2)}(0) = \frac{2\pi}{k^2}\mathrm{Im}\sum_l (2l+1)C_l^{(2)}e^{i\phi_l^{(2)}} \tag{40}$$

$$\frac{4\pi}{k}\mathrm{Im}f^{(2)}(0) = \frac{2\pi}{k^2}(-)\mathrm{Re}\sum_l (2l+1)C_l^{(2)}e^{i\phi_l^{(2)}} \tag{41}$$

$$\boxed{\sigma^{\text{decay products}} = \tfrac{4\pi}{k}(\mathrm{Im}f^{(2)}(0)\mathrm{Re}\langle 1|2\rangle + \mathrm{Re}f^{(2)}(0)\mathrm{Im}\langle 1|2\rangle)} \tag{42}$$

This optical theorem for a diffraction dissociation channel holds in first order of the amplitude. It can be expected that at high energy the amplitudes are small, so that Eq. (42) describes the measured cross section to leading order.

4. CONCLUSIONS

The extension of the principle of superposition to states that are not *identical* but *similar* with short life time in fact seems to be a suitable tool towards an understanding of the appearance of large cross sections from small amplitudes.

Before a theory of quasi diffractive processes can be constructed on such grounds, there are a few obstacles to overcome. Spin must be included because diffractive scattering is also observed when the excitation is to a different spin.[3] A formulation for an S-matrix must be found that operates on states that are not orthogonal on each other—work to be done.

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REFERENCES

Amaldi, U., Jacob, M., and Matthiae, G. (1976). *Annals Review Nuclear Science* **26**.

Blatt, J. M. and Weisskopf, V. F. (1965). *Nuclear Physics* (1952) VIII.2,

Blum, W. (2000). Talk presented on the Workshop on Time Asymmetric Quantum Mechanics and Rigged Hilbert Space Mathematics, Clausthal (Germany) 7–11 August 2000 (unpubl.)

Blum, W. and Saller, H. (2004). Energy-momentum properties and the spin of short-lived particles. *International Journal of Theoretical Physics* **43**, 1.

Blum, W. and Saller, H. (2003). Relativistic resonances as non-orthogonal states in Hilbert space, *European Physics Journal* **C28**, 279.

Leith, D. W. G. S. (1975). SLAC Summer Institute, SLAC Report 179, Vol. 1

Roman, P. (1965). *Advanced Quantum Mechanics*, Ch. 4.

[3] We report on the spin of unstable states in Blum and Saller (2004).